

Classical Limit(s) of Quantum Field Theories

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We discuss some of the issues involved in finding classical limits for quantum fields. In particular we focus on the Hamiltonian classical field and the hydrodynamic and thermodynamic limits.

1. INTRODUCTION

The aim of this paper, as can be seen from the title, is to give a general presentation of the issues involved in the effort to understand the emergence of classical behavior from quantum fields. The tentative plural in the word “classical limit” means to reflect our uncertainty in deciding in terms of which variables this notion of limit is to be understood.

There has been an impressive range of activity in the last 10–15 years on the issue of emergent classicality. The key ideas of decoherence, coarse-graining, and noise have provided the basic conceptual tools for the study of a large variety of systems, mainly in the “everyday” nonrelativistic domain. In simple models, programs of environment-induced decoherence [1, 2] and consistent histories [3–7] have been successfully implemented and manifoldly increased our understanding of the classicalization process.

Still, it is not unfair to say that similar explorations in the territory of relativistic quantum field theory have not been able to provide a concrete cartography of the terrain. Here I give a (personal and by no means exhaustive) summary of ideas, concepts, and ongoing programs in this direction.

The conceptual importance of understanding the classical limit cannot be overstated. One of the main motivation in such an undertaking has been its importance for cosmology. Somehow, the primordial quantum fluctuations have to become classical and amplified during the inflation period if the

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seeds of structure formation are to appear. In a more general context, we would need to know how the late-universe, classical hydrodynamic description arises from the highly quantum behaviour of the early epoch.

But this issue is important even outside the cosmological context. It is closely connected to the measurement problem of quantum fields, that is, what are the observables that are actually measured in a field (amplitudes, particles) and, perhaps more importantly, under what conditions does a quantum state exhibit particlelike behavior. Finally, we should remark the importance of understanding the classical limit for formulating nonequilibrium thermodynamic properties of quantum fields.

Three particular issues are addressed in this discussion.

1. General ideas on how the classical domain is identified, and the particular problems encountered in the quantum field-theoretic context.

2. Phase space classicality, that is, how and under what conditions Hamiltonian mechanics can be obtained as the limit of a quantum theory.

3. The development of a framework of ideas seeking to understand classicality through the study of hydrodynamic or thermodynamic types of coarse-graining.

2. WHAT IS INVOLVED IN CLASSICALIZATION?

By inverting the question in the title of this section, we could equally well ask what is peculiar to the quantum theory, for these features are to be suppressed when desiring a quasiclassical domain.

Decoherence. The foremost characteristic of the quantum world is coherence: the appearance of superpositions as manifested in the two-slit experiment. Decoherence then is to be defined as a return to classical statistics.

Quantum mechanics is *not* a classical probabilistic theory, especially when time evolution is taken into account. This has been highlighted in the histories approach to quantum mechanics: if a probability measure is to be defined on sets of histories such that it satisfies the Kolmogorov probability axioms a decoherence condition has to be imposed between pairs of such histories.

Alternatively, this deviation from classical statistics can be seen in the phase-space picture of quantum mechanics. For one can in various ways define n -point (unequal time) functions in phase space given an initial state and a Hamiltonian. But these fail to define a stochastic process (the failure being again the Kolmogorov axioms in another guise). Hence, the first and most important criterion for classicality is the transition from a quantum to a classical stochastic process: it is more a matter of temporal statistics rather than specific dynamics and it generically involves *coarse-graining*: focusing on a less specific information than the quantum theory allows.

Approximate Determinism. Classical mechanics is a deterministic theory, and while determinism is probably too much to demand from an approximate theory nowadays (after the advent of chaos), still it is important to clarify when Newtonian mechanics is obtained as a limit of the quantum theory. For a more restricted context, then, after decoherence one demands approximate determinism, i.e., the existence of a deterministic set of equations governing to some extent the evolution of certain observables in our physical system.

As we said, decoherence necessitates coarse-graining, which has the additional effect of introducing uncertainty, hence noise in our computations. Hence classical behavior arises at the interface of two competing requirements: too little coarse-graining and one loses classical statistics, too much and one loses predictability.

Locality. Since Einstein locality is a symmetry of the classical world and is necessary in realistic field theories, one should demand its preservation in the quasiclassical regime. The danger here lies in the EPR type of correlations present in any quantum mechanical system. Indeed it has been established that nonlocal quantum correlations are generic in quantum field theories: in a given state and for each local observable there will always be another one in a spacelike-separated region almost totally entangled with it.

This problem in some form or another had been recognized from the birth of quantum field theory. It is necessary that we seek some mechanism suppressing the EPR nonlocalities, at least when looking for a quasiclassical domain. There is little work on this subject and we refer to it mainly as an important point of unease.

3. MECHANISMS FOR CLASSICALITY

The next question is then: How does classicality come about? Given a particular system, what is the constraint enforcing it to behave classically? The answer is that classicality can be (predominantly) from causes either extrinsic or intrinsic to the system.

Extrinsic Causes. This case essentially covers what is known in the literature as the environment-induced superselection rules program. That is, the stochastic action of an environment on the system can rapidly cause decoherence. This is manifested by suppression of interferences in a basis determined primarily from the system's coupling to the environment and the consequent (approximate) diagonalization of the density matrix in that basis. This implies classicalization for the statistic associated to a diagonal density matrix (in the same basis at all times) are essentially classical.

The standard paradigm for this kind of classicalization is given by the quantum Brownian motion models [9–13]. The classical limit obtained in

this case corresponds to a particle undergoing classical (dissipative) evolution under the action of a stochastic force, the strength of which depends on the state of the environment [13, 7, 15, 16]. But we should stress that it is not true that all environments are robust enough to produce decoherence [11, 14]: typically the energy carried by the environment should be quite large (e.g., a thermal state) compared to the energy scales associated with a system. For the field theory case see refs. 17 and 18.

In the field theory case a manifestation of this phenomenon could be a tendency of an interacting field's density matrix to diagonalize in approximate eigenstates of its interaction currents [19], as, for instance, might be the case for the electron field in QED when the photonic vacuum is taken as the environment.

However, particularly in a cosmological context, there seems to be a conceptual problem into applying this line of ideas. It is difficult to conceive of an objective split between system and environment in such a case—the universe is after all a closed system, and quantum coherence (defined in terms of correlations and relative phases) is never truly lost. There is much to be said for the idea that the ever-present gravitational field is the agent for matter's decoherence [20], but what then is the agent of its own decoherence?

Intrinsic Causes. This case, which is much more difficult to analyze, can best be stated in terms of a history language: due to particular dynamics and for a large class of initial states, some coarse-grained histories are realized with probability very close to one. Hence the classicality in such a case is more a result of an approximate determinism intrinsic in the configuration of a physical system.

With regard to the question of what types of coarse-grained histories exhibit this type of behavior, we give two main examples:

1. *Phase space histories* which when suitably coarse grained essentially allow us to reproduce a classical mechanics quasiclassical domain.
2. *Histories of hydrodynamics quantities* like energy or particle density, which are more suitable for systems with a large number of degrees of freedom. They involve a much larger degree of coarse-graining, which in principle would allow for decoherence even if the underlying phase-space quantum evolution does not become classicalized.

A typical example would be the behavior of histories in a many-body system corresponding to a quantity $N(p, q)$, the number density in the single-particle phase space. The study of such histories should under general conditions enable us to derive a correlation according to some form of Boltzmann equation, which again would be the basis of a hydrodynamic description (derivation of a Navier–Stokes type of equation).

This is a highly difficult problem, essentially deriving Boltzmann's equation from the Schrödinger one. But this type of coarse-graining seems very natural in a field-theoretic context, for hydrodynamic variables being of a bulk type seem to respect more the spacetime character of a quantum field.

The separation between intrinsic and extrinsic type of classicality is essentially artificial; it corresponds to the question, "What is the correct coarse-graining from which to proceed in the derivation of the classical limit?"² We believe that for quantum fields, themselves not being localized objects, the notion of coarse-graining with respect to a separation of system and environment is rather artificial and cannot be viewed as corresponding to a generic situation. Therefore, in the rest of the paper we shall concentrate on the ideas and techniques current in the study of intrinsic types of emergent classicality.

4. PHASE-SPACE COARSE-GRAINING

The phase-space structure is encoded into the quantum theory by virtue of the canonical commutation relations

$$[\hat{Q}, \hat{P}] = i\hbar \tag{4.1}$$

A concrete realization of these operators enables the definition of a presentation of the canonical groups with elements

$$\hat{U}(q, p) = \exp(ip\hat{Q} + iq\hat{P}) \tag{4.2}$$

through which one can construct the coherent states

$$|qp\rangle = \hat{U}(q, p)|0\rangle \tag{4.3}$$

conveniently taking $|0\rangle$ as a minimum-uncertainty state (or sometimes the lowest-lying state of the Hamiltonian). They define a map of the classical phase space of the system into the Hilbert space of the quantum theory. In this classical space, a metric is naturally defined as "a classical shadow of quantum geometry"

$$ds^2 = \langle qp|d^2|qp\rangle - |\langle qp|s|qp\rangle|^2 \tag{4.4}$$

This is important: it provides a length scale with respect to which one can precisely define degrees of phase-space coarse-grainings. Taking this into account, there are two complementary ways to proceed in the study of phase-space classicality.

²This is not a new question. It was noted early that in QFT the limits $c \rightarrow \infty$ and $\hbar \rightarrow 0$ do not commute; rather, the result depends on the states upon which this limit is taken. Hence the limit could either be a particle system or a classical field.

We can proceed focusing on the evolution of the state and examine the way it remains concentrated on a phase region. Mathematically this is best achieved through the use of a quasiprojector: a positive-operator-valued measure on phase space. Given a sufficiently large and regular³ phase-space cell C , one can construct an approximate projection operator P_C such that its range consists of states well localized within C . A convenient construction is through the coherent states as

$$\hat{P}_C = \int \frac{dq dp}{2\pi\hbar} |qp\rangle\langle qp| \quad (4.5)$$

Hence determination of classicality can be viewed as establishing whether a quantum state during its time evolution remains an approximate eigenstate of phase-space projectors on cells that are correlated according to the classical equations of motion.

The above context is a bit restrictive in the sense that only quasideterministic phase-space classicality can easily be discerned. For the more general case it is good to return to the ideas briefly discussed in Section 2. One can generally define operators $\Delta(\xi)$ that correspond to “projections” onto phase-space points $\xi = (q, p)$. There are several ways to define these, but the most convenient would be to take either the Fourier transform of the operator (4.2) to get contact with the Wigner representation, or a projector to a coherent state projector to be more in line with the consistent histories approach.

Then one can define the Heisenberg-picture operators (or the corresponding entities in an open system)

$$\Delta(\xi, t) = e^{iHt} \Delta(\xi) e^{-iHt} \quad (4.6)$$

and from these the unequal-time phase-space n -point functions as [21]

$$p_n(\xi_1, t_1; \xi_2, t_2 \dots \xi_n, t_n) = \text{Tr}(\rho_0 \Delta(\xi_1, t_1) \dots \Delta(\xi_n, t_n)) \quad (4.7)$$

The hierarchy formed by the set of all these functions (or, more conveniently, their real-valued symmetrized versions) does not satisfy the Kolmogorov axioms defining a stochastic process. But it is possible that when acting with a smearing operation on scales larger than a characteristic length of the metric, one can have an approximate satisfaction of these conditions. Hence, according to this idea of coarse-graining, it is meaningful to ask when this quantum process is close to a stochastic one.

A criterion can be given through the use of information measures [23]. A most convenient is the difference $I - S$ between the Shannon–Wehrl (SW)

³That is, having a boundary sufficiently smooth so that its curvature is much larger than the characteristic length scale of the metric.

entropy [22] and the von Neumann entropy of the time-evolved state. The SW entropy is defined as the Shannon information of the probability distribution

$$p_\rho(q, p) = \langle qp | \rho | qp \rangle \tag{4.8}$$

that is, as

$$I[\rho] = - \int \frac{dq dp}{2\pi\hbar} p_\rho(q, p) \log p_\rho(q, p) \tag{4.9}$$

The SW entropy satisfies two important inequalities

$$I[\rho] \geq 1 \tag{4.10}$$

$$I[\rho] \geq S \tag{4.11}$$

A state ρ can then be said to exhibit phase-space classicality if $I - S$ remains of the order of unity during time evolution. The argument for this runs as follows: The Shannon information for a state with respect to some basis is equal to the von Neumann entropy if the density matrix is diagonal on this particular basis. Now, the coherent states form an overcomplete basis on the Hilbert space of the system. A complete orthonormal basis can be constructed from a subset of coherent states by taking a lattice on phase space with separation less than a critical value [24]. Continuity arguments suffice to show that the Shannon information with respect to the phase-space lattice basis is equal to the SW entropy up to terms of order less than unity. Hence a small value of $I-S$ at all times is a guarantee that the state remains approximately diagonal in a phase-space basis during time evolution, itself implying approximately classical statistics.

An interesting example is the case where the dynamics forces the evolution of a Gaussian state into highly squeezed state $|r(t), \phi(t)\rangle$ as, for example, in the case of a classically chaotic system in the Gaussian approximation. Then $S = 0$ and

$$I[\rho] = 1 + \log \cosh r(t) \tag{4.12}$$

implying an eventual breakdown of classicality.

Both of the above criteria can be readily translated into field theory language. The information criterion seems more suitable: coherent states can be defined even in systems with infinite number of degrees of freedom, even though in interacting theories they are generically not Gaussians. An important case is that of a scalar field evolving in a de Sitter spacetime, within the context of inflationary models. The SW entropy then increases asymptotically as $I[\rho] \simeq Ht$, pointing to the nonexistence of phase-space classicality. Since in general in cosmological spacetimes the driving term of the changing scale

factor tends generically to cause squeezing, phase-space classicality can arise only through the consideration of interaction terms.

There are also two important remarks one should make on this point:

1. Decoherence is generically a nonperturbative phenomenon, meaning that it cannot be decided by consideration of perturbations around a quadratic potential. This can be shown by an analysis of the Gaussian approximation for either open or closed systems [5, 16]. At least the knowledge of the nonperturbative classical solutions is necessary if we are to decide whether indeed classicality is to emerge. The reason for this is that the relevant object in decoherence considerations is the evolution operator e^{-iHt} (rather than the S matrix when considering scattering processes) and it is well known that its perturbative evaluation breaks down at relatively early times

2. Phase-space classicality is a minimal coarse graining that gives *robust* classicality. For instance, configuration-space histories when decohering provide in general a good indication of phase-space classicality; this is not true when time evolution involves extreme squeezing: even small perturbations are sufficient then to destroy any notion of classicality in configuration space. If one, for instance, considers unequal-time n -point functions in configuration space, they always give rise to a stochastic process [25, 26]. The full quantum mechanical features arise only when considering a sample space of quantum mechanically noncommuting observables.

5. HYDRODYNAMIC COARSE-GRAININGS

Systems like the classically chaotic ones are not expected to exhibit classical behavior at the phase-space level (unless they are coupled to a decohering environment). Some many-body systems fall within this category. But even when they do, it is generally more profitable to consider histories of much more coarse-grained quantities that have a physical significance. These are for convenience labeled as hydrodynamic variables [6, 27–29], for it is these variables classicalized that one would like to recover in order to have an effective description of such systems.

If in the environment-induced decoherence the Brownian motion has served as a typical example for the system–environment split, in this case the other fundamental paradigm in nonequilibrium statistical mechanics is expected to play an important role: the Boltzmann type of coarse-graining involving essentially treating the higher order correlations of the system as unnecessary to an effective description and considering them as a source of noise able to cause decoherence to our preferred quantities, typically densities (particle, energy, charge, etc).

There are two important ways to tackle the problem and these are briefly described in what follows.

5.1. Approximate Conservation = Approximate Decoherence

Hydrodynamic variables are usually of the density type. Energy and momentum densities are elements of the stress-energy tensor and as such they are quite important when one wants to study, for instance, the backreaction of quantum fields onto the classical Einstein equation. Hence it is very important to be able to identify quasiclassical equations of motion for these observables.

There exists a heuristic argument [27] for why these objects when smeared in a volume of order V are expected to decohere. Consider any density ρ corresponding to a conserved quantity. By virtue of the continuity equation we should have

$$\frac{\partial}{\partial t} \int_V d^3x \rho = - \int_V d^3x \nabla \mathbf{J} = - \int_{\partial V} d^2\mathbf{x} \mathbf{J} \quad (5.1)$$

In a sufficiently large volume of order L^3 the rate of change of the smeared densities will scale as L^2 rather than L^3 as with smeared densities corresponding to nonconserved currents. Given the fact that exactly conserved quantities decohere, it is reasonable to assume from continuity arguments that the same will be true approximately for approximately conserved ones.

The above argument is very general and a more careful treatment ought to put restrictions on the initial state of the field. An interesting result in this direction is the fact that decoherence follows automatically when the initial state is one of local equilibrium [29]. But such states are mixed, while in a closed system we shall generically have to work with open states. One could then say that whenever the state of a system is operationally close to one of local equilibrium,⁴ decoherence of the corresponding hydrodynamic variables is to follow.

Still, it is more of interest to know when the approach to local equilibrium even in an operational sense is a generic feature of quantum mechanical many-body systems or if it itself necessitates a sufficiently local initial state. In general, could we expect a superposition of two localized states to evolve toward a local equilibrium one? Perhaps the ignored degrees of freedom taken as an environment might be able to produce decoherence, as is the argument in the context of a simple model in ref. 30.

5.2. Correlation Histories

Let us recall the type of coarse-graining employed in the classical derivation of the Boltzmann equation. In an N -particle system the phase space is $R^6 N$ and what we are interested in is an effective equation for a distribution

⁴This is to be taken as meaning to have the same expectation values when evaluating the relevant Heisenberg-picture projector on it.

function on a *one-particle* phase space R^6 . In the process of coarse-graining one is essentially ignoring all correlation functions of degree higher than one.

More generally in many-body systems the classical description by the Liouville equation is equivalent to the BBKGY hierarchy giving a set of equations describing the evolution of the n -point functions of the system. The standard practice is to truncate the hierarchy at some level (say at 2-point functions) and consider higher order correlation functions as being “slaved” to the lower order ones. This choice of coarse-graining gives naturally dissipation and noise, but is of a completely different type from the ones used in Brownian motion: the splitting of system and environment is subtle and not associated with separating different degrees of freedom.

In the case of field theory instead of the BBKGY we have the Schwinger–Dyson hierarchy of the n -point functions. In that case truncation at the $n = 1$ level corresponds essentially to mean-field theory (mean field is what can be measured by a local observer), while at the $n = 2$ level it is essentially the Gaussian approximation. Ways of effecting this truncation have been developed using either functional methods in the closed time path (CTP) formalism [31] or in a canonical framework [32] (see also ref. 33).

There are two points one should stress concerning this approach. First, it is in some sense complementary to the search for decoherence of hydrodynamic variables explained earlier. Recall that the Boltzmann equation for a nonrelativistic many-body system contains sufficient information to allow for a derivation of the Navier–Stokes hydrodynamic equation. Of course, the important point is the identification of the conditions under which some kind of classical field-theoretic transport equation does emerge as a meaningful approximation from quantum field theory. Some of the results obtained suggest that classical behavior at this level is possible only for a restricted class of states: for instance, the ones close to the vacuum.

The other important point is that the two-point functions contain sufficient information to capture the behavior of really hydrodynamic quantities such as the energy-momentum tensor (at least its dominant part in a perturbative calculation). Hence focusing on the classicalization of lower point function correlation histories might provide an alternative approach toward obtaining classical hydrodynamic equations.

REFERENCES

- [1] W. H. Zurek, *Phys. Rev D* **24**, 1516 (1981); *Phys. Rev. D* **26**, 1862 (1982); *Prog. Theor. Phys.* **89**, 281 (1993).
- [2] E. Joos and J. D. Zeh, *Z. Phys. B* **59**, 223 (1985).
- [3] R. B. Griffiths, *J. Stat. Phys.* **36**, 219 (1984).
- [4] R. Omnès, *J. Stat. Phys.* **53**, 893 (1988); *J. Stat. Phys.* **53**, 957 (1988); *Rev. Mod. Phys.* **64**, 339 (1992).

- [5] R. Omnès, *J. Stat. Phys.* **57**, 357 (1989); *The Interpretation of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1994).
- [6] M. Gell-Mann and J. B. Hartle, In *Complexity, Entropy and the Physics of Information*, W. Zurek, ed. Addison-Wesley, Reading, Massachusetts, (1990); *Phys. Rev. D* **47**, 3345 (1993).
- [7] J. B. Hartle, Spacetime quantum mechanics and the quantum mechanics of spacetime, in *Proceedings of the 1992 Les Houches School, Gravitation and Quantizations* (1993).
- [8] L. G. Redhead and F. Wagner, *Found Phys. Lett.* **11**, 111 (1998).
- [9] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965); R. P. Feynman and F. L. Vernon, *Ann. Phys. (N.Y.)* **24**, 118 (1963).
- [10] A. O. Caldeira and A. J. Leggett, *Physica A* **121**, 587 (1983).
- [11] B. L. Hu, J. P. Paz, and Y. Zhang, *Phys. Rev. D* **45**, 2843 (1992); *Phys. Rev. D* **47**, 1576 (1993).
- [12] B. L. Hu and A. Maticz, *Phys. Rev. D* **49**, 6612 (1994).
- [13] W. G. Unruh and W. H. Zurek, *Phys. Rev. D* **40**, 1071 (1989).
- [14] P. M. V. Barone and A. O. Caldeira, *Phys. Rev. A* **43**, 57 (1991).
- [15] H. F. Dowker and J. J. Halliwell, *Phys. Rev. D* **46**, 1580 (1992).
- [16] C. Anastopoulos, *Phys. Rev. E* **53**, 4711 (1996).
- [17] W. H. Zurek and J. R. Anglin, *Phys. Rev. D* **53**, 7327 (1996).
- [18] S. Habib, Y. Kluger, E. Mottola, and J. P. Paz, *Phys. Rev. Lett.* **76**, 4660 (1996).
- [19] C. Anastopoulos and A. Zoupas, *Phys. Rev. D* **58** (1998).
- [20] R. Penrose, In *Quantum Concepts in Space and Time*, R. Penrose and C. J. Isham, eds. (Clarendon Press, Oxford, 1986); F. Karolyhazy, A. Frenkel, and B. Lukacs, In *Quantum Concepts in Space and Time*, R. Penrose and C. J. Isham, eds. (Clarendon Press, Oxford, 1986); C. Percival, quant-ph 9508021; C. Anastopoulos, *Phys. Rev. D* **54**, 1600 (1996); B. S. Kay, hep-th/9810077.
- [21] M. D. Srinivas, *Phys. Rev. D* **15**, 2837 (1977).
- [22] A. Wehrl, *Rev. Mod. Phys.* **50**, 221 (1978), A. Wehrl, *Rep. Math. Phys.* **16**, 353 (1979).
- [23] C. Anastopoulos, quant-ph/9805055.
- [24] A. M. Perelomov, *Generalized Coherent States and Their Applications*, (Springer-Verlag, Berlin, 1986).
- [25] E. Nelson, *Phys. Rev.* **150**, 1079 (1966).
- [26] F. Guerra, *Phys. Rev.* **77**, 263 (1981).
- [27] J. B. Hartle, R. Laflamme, and D. Marolf, *Phys. Rev. D* **51**, 7007 (1995).
- [28] T. Brun and J. J. Halliwell, *Phys. Rev. D* **54**, 2899 (1996).
- [29] J. J. Halliwell, quant-ph 9805062.
- [30] E. Calzetta and B. L. Hu, quant-ph/9809084.
- [31] E. Calzetta and B. L. Hu, In *Directions in General Relativity, Vol. II: Brill Festschrift*, B. L. Hu and T. A. Jacobson, eds. (Cambridge University Press, Cambridge, 1994); in *Heat Kernel Techniques and Quantum Gravity*, S. A. Fulling, ed. (Texas A & M Press, College Station, Texas, 1995); gr-qc 9501040.
- [32] C. Anastopoulos, *Phys. Rev. D* **56**, 1009 (1997).
- [33] F. Cooper, S. Habib, Y. Kluger, and E. Mottola, *Phys. Rev. D* **55**, 6471 (1997).